Attention is called to the following two papers, one by R. K. Moore, the other by W. L. Pritchard and J. A. Mullen. Since the papers are very closely related, the objection might arise that either one or the other should be published, but not both. However, the editorial board felt that both papers are deserving of considerable credit, and indeed the subject itself is of great interest, not only to microwave people, but to systems people as well.

Furthermore, although Mr. Moore's paper was received by the Editor of these TRANSACTIONS on August 7, 1956, whereas Messrs. Pritchard and Mullen's paper was not received until October 1, 1956, it is only fair to point out that the latter paper had been submitted to the Editor of PROCEEDINGS some time prior to September of last year.—*The Editor*

The Effects of Reflections from Randomly Spaced Discontinuities in Transmission Lines*

RICHARD K. MOORE†

Summary—**Reflections from randomly spaced transmission line discontinuities can cause serious attenuation and distortion of pulses in the lines, and the presence of reflections at the sending end may be undesirable. The effect of these discontinuities may be described in terms of the mathematics for combining outputs from oscillators with random frequencies. The location of the discontinuity corresponds to the frequency of an oscillator. The phase constant of the transmission line is analogous to time for the oscillators. Use of spectrum and filter analogies permits approximate determination of discontinuity locations from measurements. Use of known space and size distributions permits statistical prediction of attenuation and of size of reflected wave at the sending end.**

TRANSMISSION lines often contain small discontinuities which may affect their operation—either by causing undesired reflections at the input, or by increasing the attenuation over that which would be present without the discontinuities.[1–4] These discontinuities may be regularly spaced due to such things as beads supporting center conductors for coaxial transmission lines, couplings between individual sections of waveguide, or bends associated with the way in which a coaxial line was rolled up during storage. On the other hand, they may be randomly spaced due to such things as random breakage during manufacture, dents in outer conductors of solid coaxial lines or in waveguide, random migration of the center conductor in coaxial lines, either flexible or solid, and pinching of a coaxial line by various causes. In this paper, expressions are developed for the additional attenuation and reflection due to randomly spaced small discontinuities.

For randomly spaced discontinuities, a statistical relation has been established between the reflection or attenuation-vs-frequency curve and the spacing of discontinuities. It is shown that the mathematics for reflections from randomly spaced discontinuities is the same as that for the combination of large numbers of oscillators with random frequencies. The technique of analysis used to obtain the frequency spectrum for the oscillators has been used here to locate the discontinuities. Our analogy compares the frequency of the individual oscillator with the coordinate (on the transmission line) of the individual discontinuity. The time variable, in the case of oscillators, is analogous to the phase constant of the transmission line.

The methods shown may be used to calculate the effects of discontinuities on attenuation and reflection if the sizes and magnitude of the discontinuities are known. They may also be used to ascertain, as nearly as possible, the size and location of discontinuities by using a measurement of the attenuation or reflection as a function of frequency. Because this measurement may be made over only a limited range of frequencies in any practical case, the exact location and number of discontinuities cannot be specified. Rather, a spectrum may be specified which gives the best idea as to the size and location of discontinuities which can be obtained with any specified frequency range for the measurement.

Fig. 1 represents a cable with discontinuities spaced a distance Δx_i apart along its entire length. The reflec-

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[1] D. A. Alsberg, "More on the sweep-frequency response of RG/6U cable," PROC. IRE, vol. 41, p. 936; July, 1953.

[2] W. T. Blackband, "The sweep-frequency response of RG/6U," PROC. IRE, vol. 40, pp. 995–996; August, 1952.

[3] D. A. Alsberg, "A precise sweep-frequency method of vector impedance measurement," PROC. IRE, vol. 49, pp. 1393–1400; November, 1951.

[4] W. T. Blackband and D. R. Brown, "The two-point method of measuring characteristic impedance and attenuation of cables at 3,000 mc," *J. IEE*, vol. 93, Part IIIA, pp. 1383–1386; September, 1946.

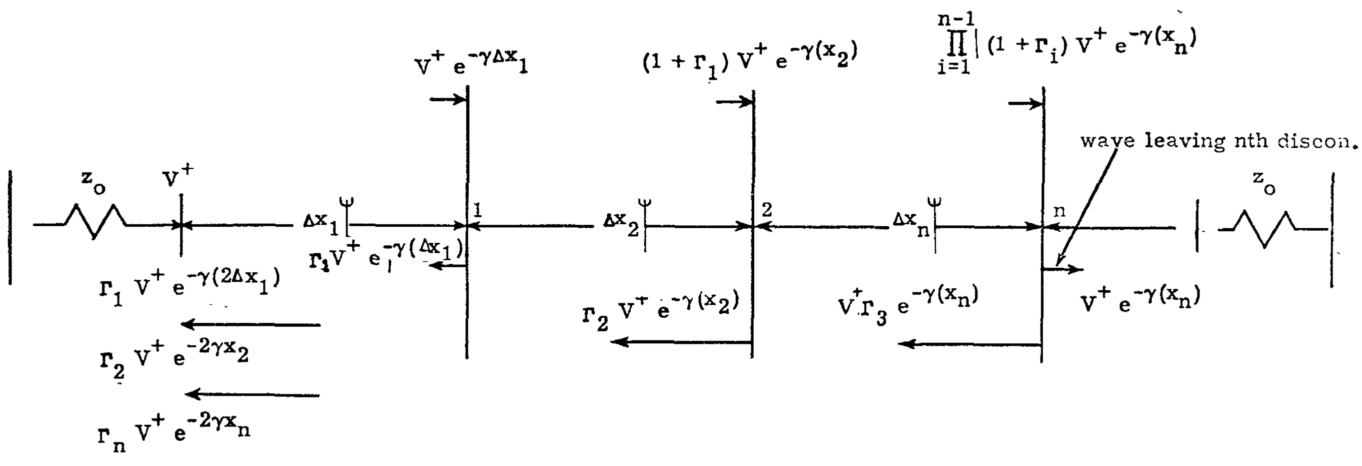

Fig. 1.

tion coefficient Γ_i for voltage is assumed to be small so that only first-order reflections need be considered; *i.e.*, $|\Gamma_i| \ll 1$.

In the figure and subsequent discussion,

V^+ = incident input;
V^- = reflected wave at input;
V_i = total input = $V^+ + V^-$;
V_t = total output;
$\gamma = \alpha + j\beta$, the propagation constant;
L = line length;
n = index of discontinuities;
N = total number of discontinuities;
Γ = voltage reflection coefficient.

The voltage reflected back to the origin of the line is given, in the notation of this paper, by

$$V^- = \sum_{n=1}^{N} (V^+ e^{-\alpha x_n} \Gamma_n e^{-j\beta x_n})(e^{-\alpha x_n} e^{-j\beta x_n}). \tag{1}$$

Here, the coordinate of the nth reflecting discontinuity is given by x_n. This equation is made up in the following manner. The first bracket represents the magnitude and phase of the reflected wave at the nth discontinuity. The magnitude of the incident wave at the origin is given by V^+. The first exponential gives the effect of attenuation on the line from the origin to the point of discontinuity. Γ_n represents the magnitude of the voltage reflection coefficient, and the second exponential term represents the phase shift down the line from the origin to the nth discontinuity. The expression in the second bracket shows the additional effect of transmitting this wave from the reflection point back to the origin, with both phase and amplitude terms. In this equation only first-order reflections are considered. If $|\Gamma|$ is large enough, second-order effects may be significant; so the analysis here only applies to small $|\Gamma|$.

Note also that the effect on the transmitted wave has been neglected. In Fig. 1, this effect is shown at the top but not at the bottom.

To simplify the form of this expression for calculating the probable amplitude of the reflected wave, the attenuation factors and the reflection coefficients have been combined in one symbol, A, and the two phase factors have also been combined. The result is given by

$$V^- = V^+ \sum_{n=1}^{N} A_n e^{-j2\beta x_n}. \tag{2}$$

If we assume random spacing for the discontinuities so that the principal value of the phase term ($2\beta x_n$) may with equal likelihood take on any value between zero and 2π, then we may treat the resulting sum in the same way that one would treat the result of combining the outputs of a large number of oscillators having frequency $2x_n$ and time variation as $(-\beta)$.

When the conditions stated in the preceding paragraphs are met, whether it be for reflections from randomly spaced discontinuities or for waves from oscillators of various frequencies, the mean square value of the resultant vector obtained by summing up all the components is given by the sum of the squares of the various component vectors. This is indicated by

$$\overline{|V^-|^2} = (V^+)^2 \sum_{n=1}^{N} A_n^2. \tag{3}$$

For time variations, rather than variations with β, this is the same as saying the average power resulting from a large number of waves of different fre quencies is equal to the sum of the power contained in the various waves. This is a well-known phenomenon and applies even though the waves are harmonically related.

Eq. (3) is an average over β, the phase propagation constant. If will be recognized that, for the transverse electromagnetic mode, β varies only as the first power of frequency and, therefore, this may also be considered as an average over frequency; thus, it might be measured by making measurements at a large number of frequencies and averaging them. In a waveguide, of course, or in an inhomogeneous medium, β may vary with some other parameter besides frequency or may vary with frequency in a more complex manner than a straight linear variation.

If one is to utilize measurements of the reflected wave or transmitted wave to determine an average discontinuity, or conversely, if one is to use an average value of discontinuity to determine the reflected voltage or transmitted voltage, it is necessary to separate the effect of the reflection coefficient from the attenuation in the line. If there is no relationship between the individual reflection coefficients and their position on the line, that is, if their variations about the average reflection coefficient are random and independent of location, one may assume

$$A_n = \overline{\Gamma} e^{-2\alpha x_n}, \tag{4}$$

where $\overline{\Gamma}$ represents the mean value of the reflection coefficient. The assumption of random distribution of the coordinates x_n means that

$$\text{Prob}\,(x < x_n < x + dx) = \frac{dx}{L}, \tag{5}$$

that is, the probability of finding x_n in any interval dx for a line of length L is the length of the interval divided by the total length of the line.

Utilizing this probability, one may compute the expected value for the mean square voltage as given by (3) by use of the relation

$$E\left(\sum_{n=1}^{N} A_n^2\right) = N \int_0^L (\overline{\Gamma})^2 e^{-4\alpha x} \frac{dx}{L} = N\overline{A^2}. \tag{6}$$

It should be noted that this represents propagation continuously down the transmission line; that is, the entire length of the line is illuminated at any one time.

Eq. (6) defines a mean square value of A which can be used in analysis of the performance of lines from measurements or in prediction of their performance based upon knowledge of discontinuities and their location. The result is given by

$$\overline{|V^-|^2} = (V^+)^2 \frac{N\overline{\Gamma}^2}{4\alpha L}(1 - e^{-4\alpha L}). \tag{7}$$

From this we can see that the magnitude of the reflected root-mean-square voltage is given by

$$V_a^- = (\overline{|V^-|^2})^{1/2} = V^+\overline{\Gamma}\sqrt{\frac{N(1-e^{-4\alpha L})}{4\alpha L}}\,. \tag{8}$$

The input voltage to the line is the sum of the incident and reflected voltages, as measured at the input. For the case discussed above, the input voltage is, therefore, given by

$$V_i = V^+\left(1 + \frac{|V^-|}{V^+} e^{j\phi}\right), \tag{9}$$

where ϕ is the phase angle of V^-.

The relationship of the phase of the reflected wave to that of the transmitted wave is important in determining the magnitude of the input voltage. The phase angle may take on with equal likelihood any value between zero and 2π, hence

$$\text{Prob}\,(\phi < \phi_n < \phi + d\phi) = \frac{d\phi}{2\pi}\,. \tag{10}$$

It should be noted that the reflected voltage has a magnitude which is known only statistically for any given β. Its variation is the same as that for the magnitude of a white noise voltage since (2) indicates that the same representation may be used for reflected voltage, with β as the variable, as is used for noise voltage where time is the variable.

The distribution function for the reflected voltage is called the Rayleigh distribution if the total number of terms in the sum of (2) approaches infinity. The Rayleigh distribution is the result of an infinite-step random walk in a plane, and this is a well-known problem of statistics. It is readily seen upon examination of the distribution functions for Rayleigh and for finite random walks that, at least in the case where all steps are equal in length, a five-step random walk gives essentially the same distribution curve, except in the extreme limits, as an infinite number of steps would give. Since, in general, one is interested here in the situation in which there is a fairly large number of discontinuities, one may without large error replace the distribution for the resultant voltage with a finite number of discontinuities by the equivalent Rayleigh distribution. The result of doing this is shown by

$$\text{Prob}\,(V < |V^-| < V + dV) = \frac{2V}{N\overline{A^2}} e^{-V^2/N\overline{A^2}} dV. \tag{11}$$

The distribution of V^- [or $(|V^-|/V^+)e^{j\phi}$ of (5)] is a two-dimensional one given by

$$P(V^-)dVd\phi = \frac{1}{2\pi} P(|V^-|)dVd\phi. \tag{12}$$

Hence

$$P\left(\frac{V_i}{V^+}\right) = P\left[\left(1 + \frac{|V^-|e^{j\phi}}{V^+}\right)\right]. \tag{13}$$

It should be noted that this is identical statistically to the problem of the combination of a large steady signal and a Rayleigh distributed signal.[5] The resulting distribution function for the magnitude, converted from Rice's notation to ours, is given by

[5] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332; July, 1944, and vol. 24, pp. 46–156; January, 1945.

$$P\left(\frac{|V_i|}{V^+}\right) = \frac{|V_i|}{V^+}\sqrt{\frac{2}{N\bar{A}^2}} \cdot \exp\left[-\frac{1+\left(\frac{|V_i|}{V^+}\right)^2}{N\bar{A}^2}\right] I_0\left(\frac{2|V_i|}{V^+N\bar{A}^2}\right), \tag{14}$$

where I_0 is the Bessel function of the first kind with imaginary argument. One may use this expression to calculate the probable loss in the transmission line. The ratio of output to input voltage of the line is given by

$$\text{Loss} = \left|\frac{V_t}{V_i}\right| = \left|\frac{V_t/V^+}{V_i/V^+}\right| = R. \tag{15}$$

The distribution for loss may be obtained from (14) by use of the identity from probability theory

$$P(x) = Q(y)\left|\frac{dy}{dx}\right| \cdot$$

If we assume that V_t does not vary with β (to a first-order approximation), the resultant loss distribution is given by

$$Q(R) = R^2 \frac{P\left(\frac{|V_i|}{V^+}\right)}{|V_t|/V^+} = \frac{R^2 P\left(\frac{|V_t|}{V^+R}\right)}{|V_t|/V^+} \cdot \tag{16}$$

Curves of $P(|V_i|/V^+)$ are given by Rice and others so that this probability may be readily plotted. Utilizing results which may be derived for V_t, we have

$$Q(R) = \frac{R^2 P\left[\frac{(1+\bar{\Gamma})^N e^{-\alpha L}}{R}\right]}{(1+\Gamma)^N e^{-\alpha L}}$$
$$= \text{Prob}\,[R < \text{Loss} < R + dR]. \tag{16a}$$

Eqs. (11) and (16a) give the probability of any amplitude of reflection or loss, respectively. Thus, if one is interested in the reflected signal, he may use (11) to find the probability of it taking on any value; and if one is interested in the attenuation to a transmitted wave, he may use (16a).

Normally, one would not be interested in a reflection if the wave filled the entire line as the case of a continuous wave. On the other hand, if a short pulse were transmitted down the line, only a portion of the line would be illuminated at any given time and a reflected signal would occur after the end of the transmitted pulse which might have some deleterious or desirable effect on equipment located at the transmitting point. When the short pulse is used, the integral of (6) must be carried out over the limits of the illuminated region as indicated in

$$E\left(\sum_{N=1}^{M} A_n^2\right) = M\int_{c(t-\tau)/2}^{ct/2} (\bar{\Gamma})^2 e^{-4\alpha x}\,\frac{dx}{(c\tau/2)} \tag{16b}$$

where M is the number of discontinuities illuminated, t is the time from start of transmission of the pulse, τ is the pulse length, and all the other quantities are the same as in (6).

The illumination of the line is illustrated in Fig. 2. At a given time during transmission of the pulse the illumination is as indicated in Fig. 2(a), but the portion contributing to the reflection at that time is shown in Fig. 2(c). Since the reflected wave must travel down and back, only the illuminated region corresponding to a *round-trip* time t will contribute. This has the same effect as if the transmission velocity were halved in the line, insofar as reflected waves are concerned. Figs. 2(b) and 2(d) show the situation where the end of the pulse has occurred (16b).

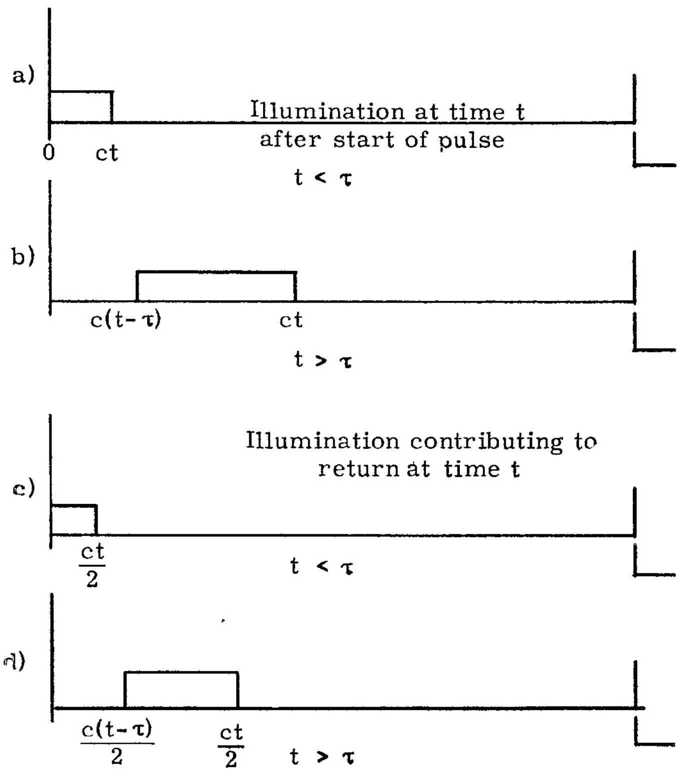

Fig. 2.

When this situation prevails, the effect of the reflected wave on the total loss is noted as a reduction in amplitude plus an effect on V_i, similar to that involved in derivation of (16), in which the reflections which occur during transmission of the pulse are the only ones which are significant. Thus, in such a case, the integration for determining V_i modification would have to be carried out no further than from zero to ½ pulse length along the line. The transmitted pulse would be distorted at different points by an amount determined by carrying this integration to an appropriate upper limit.

Information about the spacing, magnitude, and location of the discontinuities in the line can be found by studying the magnitude of the reflected wave as a function of β. In the succeeding development, it will be assumed that one is actually measuring the amplitude of the reflected wave. This can be done by using a pulse to excite a portion of the line, or it can be done by going backwards through the development which results in (16).

Consider the reflected wave as a function of β as indicated in

$$A(\beta) = \left| \sum_{n=1}^{N} A_n e^{-j2x_n\beta} \right|. \tag{17}$$

This is a quantity which may be measured, provided one knows the magnitude of the incident pulse. The autocorrelation vs β of such a measured function may be calculated; (β would be varied by varying frequency). The autocorrelation is given by

$$B(\beta) = \lim_{F\to\infty} \int_0^F |A(b)| \, |A(b+\beta)| \, db. \tag{18}$$

By itself the autocorrelation has no particular significance in this case since we are really interested in the space variation, not β variation. Typical examples of $A(\beta)$ and $B(\beta)$ are shown in Fig. 3.

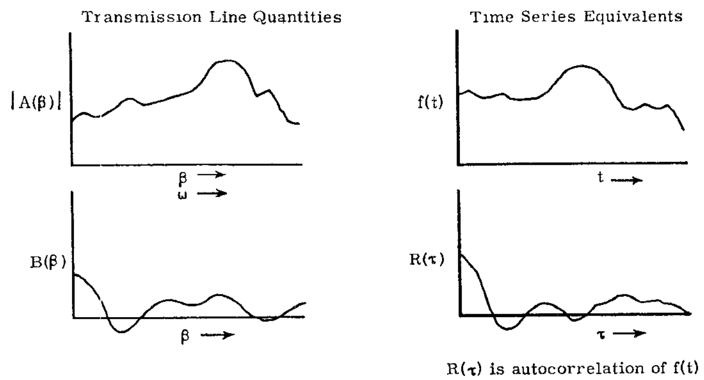

Fig. 3—Reflected wave functions and their time series analogs.

We may obtain the space variation from the autocorrelation by utilizing the relationship that the cosine transform of the autocorrelation function is the power spectrum. This is used in time series analysis quite frequently, in which case the autocorrelation vs time is computed and the transform is taken to obtain the spectrum of power as a function of frequency. In this case, we compute the autocorrelation as a function of β and transform in such a way as to obtain a power spectrum as a function of x. Actually, of course, since there are discrete discontinuities, the power spectrum of reflection vs x should contain discrete lines; and, in fact, a complete Fourier analysis would give the lines. This is indicated in Fig. 4. Since we cannot measure a complete range of β, it is not possible to obtain the exact locations and magnitudes of the discontinuities.

If, instead of a number of discrete discontinuities, there were a continuum of small discontinuities, then the spectrum would itself be continuous as a function of distance. In that case, the magnitude of the spectrum would be the reflected power within a length dx at a distance x from the origin. Because of our incomplete information (due to lack of an infinite range in frequency), such a continuous spectrum will be obtained anyway, and it will give some indication of the location of the discrete reflectors. Fig. 5 illustrates this effect, and the various quantities used in describing it.

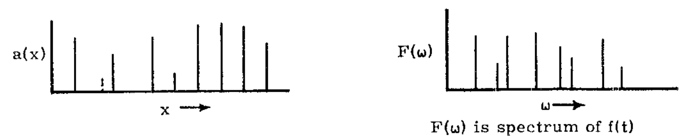

Fig. 4—Discontinuities and time series analogs.

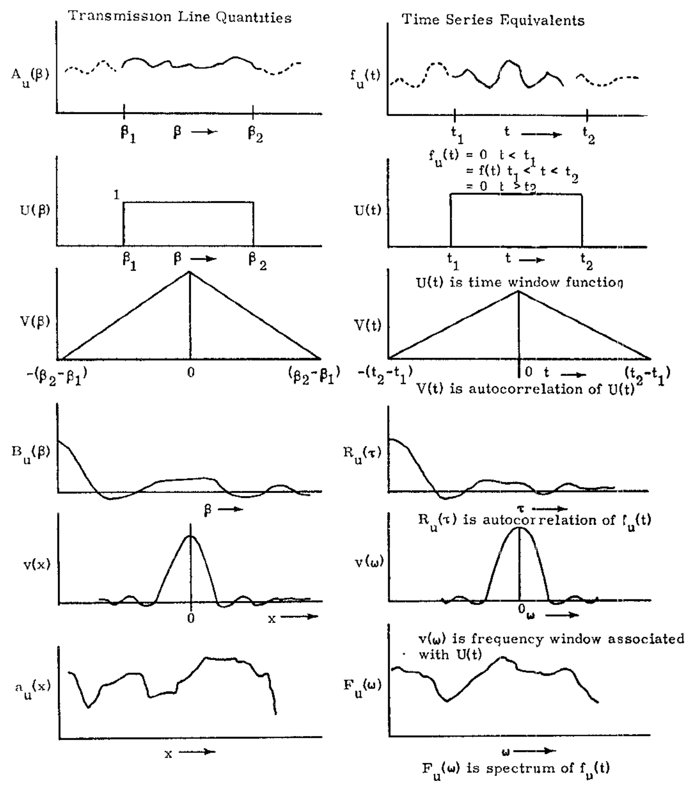

Fig. 5—Effect of finite frequency range of measurement on inferred distance spectrum of discontinuities, and its time-series analog.

The expression for determining the spectrum is given by

$$a(x) = 4 \int_0^\infty A(\beta) \cos x\beta d\beta. \tag{19}$$

Instead of the reflection function A, we will actually use a function A_u which is measured over the range β_1 to β_2, since it would be impossible to make measurements over the entire range of β (or frequency).

$$\begin{aligned} A_u(\beta) &= 0, & \beta &< \beta_1 \\ &= A(\beta), & \beta_1 &< \beta < \beta_2 \\ &= 0, & \beta_2 &< \beta. \end{aligned} \tag{20}$$

Associated with this will be an autocorrelation function

$$B_u(\beta) = \int_0^\infty | A_u(b) | \, | A_u(b + \beta) | \, db \tag{21}$$

and associated with the autocorrelation function will be a modified space power spectrum given by

$$a_u(x) = 4 \int_0^\infty B_u(\beta) \cos x\beta d\beta. \tag{22}$$

It is possible to write B_u in terms of A instead of A_u by introducing a function:

$$\begin{aligned} U(\beta) &= 0, & \beta &< \beta_1 \\ &= 1, & \beta_1 &< \beta < \beta_2 \\ &= 0, & \beta_2 &< \beta. \end{aligned}$$

When this is done, we have B_u

$$B_u(\beta) = \int_0^\infty [| A(b) | \, | A(\beta + b) |][U(b)U(b + \beta)]db. \tag{21a}$$

A is a stochastic variable and U may be considered as an independent stochastic variable; that is, the probability of an occurrence of one should have no effect on the probability of occurrence of the other. If we assume this sort of independence of A's and U's, we may utilize a theorem of probability which states that

$$\overline{xy} = \bar{x}\,\bar{y}. \tag{23}$$

The result of application of this theorem to (21a) is

$$\begin{aligned} B_u(\beta) &= \int_0^\infty | A(b) | \, | A(b+\beta) | \, db \int_0^\infty U(b)U(b+\beta)db \\ &= B(\beta)V(\beta), \end{aligned} \tag{21b}$$

where V is a triangular window function associated with the autocorrelation and given by

$$\begin{aligned} V(\beta) &= 0, & \beta &< -(\beta_2 - \beta_1) \\ &= (1 + \beta)(\beta_2 - \beta_1), & -(\beta_2 - \beta_1) &< \beta < 0 \\ &= (1 - \beta)(\beta_2 - \beta_1), & 0 &< \beta < (\beta_2 - \beta_1) \\ &= 0. & (\beta_2 - \beta_1) &< \beta. \end{aligned} \tag{21c}$$

In this notation, (22) becomes

$$a_u(x) = 4 \int_0^\infty B(\beta)V(\beta) \cos x\beta d\beta, \tag{22a}$$

and a_u may also be given in terms of the convolution of a and v as

$$a_u(x) = a(x)_* v(x) \tag{24}$$

where v is the transform of V and is given by

$$v(x) = \frac{\beta_2 - \beta_1}{2} \left[\frac{\sin x\left(\frac{\beta_2 - \beta_1}{4}\right)}{x\left(\frac{\beta_2 - \beta_1}{4}\right)} \right]^2. \tag{25}$$

The physical meaning of this is that the lines of the spectrum, a, are looked at through a space window or a space filter v, which has the form of $(\sin ax/x)^2$. This means that, instead of the discrete lines, one sees each line spread out to the shape of v. If the lines are far enough apart in the power spectrum (that is, if the discontinuities are far enough apart), then the spectrum measured will be a succession of these window functions and one can easily locate the individual lines and their magnitudes and, hence, the individual discontinuities and their magnitude. On the other hand, if the discontinuities are fairly close together, there will be a jumbling of these responses to the various discontinuities and it will not be possible to determine the exact location and magnitude of the individual discontinuities but only the average distribution of discontinuities and their average magnitude at any given spot. This corresponds closely to the time function situation in which one is looking for discrete lines in a frequency spectrum but is forced to take an inaccurate determination of these lines due to the fact that he must use a finite sample rather than observing for an infinite period of time.

Using this method of analysis on the reflected pulse should make it possible to determine without too much difficulty the location and size of discontinuities. The application of this method to the case where loss is measured will be somewhat harder because of the complications introduced by going through the process of (15), but still should be quite feasible.

It has been shown that distribution functions of the values for reflected and transmitted waves on a transmission line with randomly spaced discontinuities may be computed if something is known of the probability distribution, average size, and number of the discontinuities. Conversely, it is shown that a measurement of transmission or reflection as a function of frequency may be used to locate approximately the discontinuities causing the reflection.

The techniques illustrated here should be useful both in determining the performance of known transmission lines and in establishing the mechanism and location for reflections in transmission lines whose discontinuous properties are not known.

ACKNOWLEDGEMENT

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